# An observation and perhaps an interesting conclusion on continued fractions

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Few months ago, when tutoring a student on SAT math I ran into an exercise that involved what is known as continued fractions. Obviously, that was not the first time I saw them, but encountered them again relatively recent in 2006 when studying for my MA in Math degree. It's one of those many things you see, appear interesting or not and you move on or forget about. Then, you run into them again, and perhaps with a different mindset you start "playing" a bit with them and sometimes find something intriguing, at least at the beginning. Then, you get more into it and find more and more, and so on.

So, it's obvious that 3/2 can be written as:

$$\mathbf{3/2} = 1 + \frac{1}{2}$$

Then, you try "to go deeper" with that (continued) fractions-like representation and:

For $\mathbf{5/3} = 1 + \cfrac{1}{1 + \cfrac{1}{2}}$

Next:

$$\mathbf{8/5} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

$$\mathbf{13/8} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}$$

and so on…

Next, obviously you observe that numbers start alternatively repeating themselves as nominators and denominators.

**So, you conclude that if you start with the fraction k/p, and follow the continued fraction-like representation, the next one is going to be (p+k)/k. (1)**

Obviously, like any true mathematician, as exciting this observation can be, you are skeptical. So, try different numbers (fractions).

I did so by starting now with **4/3** and noticed as:

$$\mathbf{4/3} = 1 + \cfrac{1}{3}$$

then $$\mathbf{7/4} = 1 + \cfrac{1}{1 + \cfrac{1}{3}}$$

and $$\mathbf{11/7} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}$$

and so on, clearly the same pattern.

Then $$\mathbf{5/4} = 1 + \cfrac{1}{4}$$

The next is $$\mathbf{9/5} = 1 + \cfrac{1}{1 + \cfrac{1}{4}}$$

And $$\mathbf{14/9} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}$$

**So, it looks like if you start with a fraction like (n+1)/n, you can write it as an continued fraction, whereas the last denominator is n. And, in all cases, the relationship between two consecutive such terms in that sequence is the one in (1) above!**

On the other hand, if you start with a fraction like n/(n+1) is the same thing like starting with the (improper) fraction (n+1)/n and not having 1 in front before the sequence of continuous fractions.

Obviously, like any scientist or mathematician you want (are tempted) to generalize. Thus, you ask the typical question: “How about if given any fraction, s/t, proper or improper. The answer is simple, if the fraction is improper, you do the division and get out the integer part, after that the proper fraction left can still be written as a continuous fraction, repeating the pattern noticed above. Now you get excited about that finding!

Next, intrigued a bit about the repeating pattern showed in (1), obviously the Fibonacci series comes into mind! So, given a number (fraction) n=a/b, where a = b+1, is like a Fibonacci series where:

$F_0 = b$, $F_1 = a$, $F_2 = a + b$, $F_3 = 2a + b$, and so on, the k term starting with n = a/b, can be written as:

$n_k = [a*F_k + b*F_{k-1}]/[a*F_{k-1} + b*F_{k-2}]$

Now, replacing $F_k = F_{k-1} + F_{k-2}$ we get:

$n_k = [(a + b)*F_{k-1} + a*F_{k-2}]/[a*F_{k-1} + b*F_{k-2}]$ or also written as:

$n_k = [(a + b)*F_{k-1}/F_{k-2} + a]/[a*F_{k-1}/F_{k-2} + b]$

Now, for notation simplicity, let's write $f_{k-1} = F_{k-1}/F_{k-2}$ and diving by a, we have:

**$n_k = [2 f_{k-1} + 1]/[f_{k-1} + 1]$ or $n_k = 2 + 1/f_{k-1}$ (2)**

Next, let's generalize, if n=a/b, where a = b + p (all letters here are, of course integers!), then following the same computations, we arrive to:

**$n_k = p + 1/f_{k-1}$ (3)**

The final observation/conclusion is that if we keep computing terms **$n_k$** of the above sequences, and knowing that $\lim_{n\to\infty} f_{k-1} = \phi$ the Golden Ratio, thus

**$\lim_{n\to\infty} n_k = p + 1/\phi$ (4)**